

SMOOTH RATIONAL SURFACES VIOLATING KAWAMATA–VIEHWEG VANISHING

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ABSTRACT. We show that over any algebraically closed field of positive characteristic, there exists a smooth rational surface which violates Kawamata–Viehweg vanishing.

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1. INTRODUCTION

It is a well known fact that Kodaira vanishing fails in positive characteristic [Ray78]. Nevertheless, it has often been believed that a stronger version, namely Kawamata–Viehweg vanishing, holds over a smooth rational surface (e.g. see [Ter98, Xie10]).

In this note, we show that this is indeed not true:

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Theorem 1.1 (Theorem 3.1). *Let k be a field of positive characteristic. Then there exist a smooth projective rational surface X over k , a Cartier divisor D , and a \mathbb{Q} -divisor $\Delta \geq 0$ such that*

- (1) (X, Δ) is klt,
- (2) $D - (K_X + \Delta)$ is nef and big, and
- (3) $H^1(X, \mathcal{O}_X(D)) \neq 0$.

To prove Theorem 1.1, we use some surfaces constructed by Langer [Lan16]. If $k = \mathbb{F}_p$, then X can be obtained by taking the blowup of $\mathbb{P}_{\mathbb{F}_p}^2$ along all the \mathbb{F}_p -rational points. Since the proper transforms $L'_1, \dots, L'_{p^2+p+1}$ of the \mathbb{F}_p -lines L_1, \dots, L_{p^2+p+1} are pairwise disjoint, we can contract all these curves and obtain a birational morphism $g: X \rightarrow Y$ onto a klt surface Y such that $\rho(Y) = 1$ (cf. Lemma 2.4). Note that $-K_Y$ is ample if and only if $p = 2$ (cf. Lemma 2.4). Further, we show:

- For any $p > 0$, Y is obtained as a purely inseparable cover of \mathbb{P}^2 (cf. Theorem 4.1). If $p = 2$, then the morphism $Y \rightarrow \mathbb{P}^2$ is induced by the anti-canonical linear system $|-K_Y|$ (cf. Remark 4.2).
- If $p = 2$, then the Kleiman–Mori cone $NE(X)$ is generated by exactly 14 curves (cf. Theorem 5.4).
- If $p = 2$, then X is isomorphic to a surface constructed by Keel–McKernan (cf. Proposition 6.4).

Related results: After Raynaud constructed the first counterexample to Kodaira vanishing in positive characteristic [Ray78], several other people studied this problem (e.g. see [DI87], [Eke88], [SB91], [Kol96, Section II. 6], [Muk13], [DF15]). In particular, Fano varieties are known to violate Kawamata–Viehweg vanishing. As far as the authors know, the examples constructed by Lauritzen–Rao [LR97] (of dimension at least 6) are the only one over an algebraically closed field. If we admit imperfect fields, then Schröer and Maddock constructed log del Pezzo surfaces with $H^1(X, \mathcal{O}_X) \neq 0$ [Sch07, Mad16]. In [CTW16], the authors and Witaszek show that Kawamata–Viehweg vanishing holds for klt del Pezzo surfaces in large characteristic. On the other hand, if $p = 2$, then the surface mentioned above is a smooth weak del Pezzo surface (cf. Lemma 2.4), hence our result cannot be extended to characteristic two (see also Proposition A.1).

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2. PRELIMINARIES

2.1. Notation. We say that X is a *variety* over a field k , if X is an integral scheme which is separated and of finite type over k . A *curve* (resp. *surface*) is a variety of dimension one (resp. two). We say that two schemes X and Y over a field k are *k -isomorphic* if there exists an isomorphism $\theta: X \rightarrow Y$ of schemes such that both θ and θ^{-1} commutes with the structure morphisms: $X \rightarrow \operatorname{Spec} k$ and $Y \rightarrow \operatorname{Spec} k$. Given a proper morphism $f: X \rightarrow Y$ between normal varieties, we say that two \mathbb{Q} -Cartier \mathbb{Q} -divisors D_1, D_2 on X are *numerically equivalent over Y* , denoted $D_1 \equiv_f D_2$, if their difference is numerically trivial on any fibre of f .

We refer to [KM98, Section 2.3] or [Kol13, Definition 2.8] for the classical definitions of singularities (e.g. *klt*) appearing in the minimal model programme. Note that we always assume that for any klt pair (X, Δ) , the \mathbb{Q} -divisor Δ is effective.

2.2. Construction by Langer. We now recall the construction of a rational surface due to Langer [Lan16] (see also [Har77, Exercise III.10.7]). A similar method was used to construct also some K3 surfaces and Calabi–Yau threefolds (cf. [Hir99], [DK03]).

Notation 2.1. (1) Let $q := p^e$, where p is a prime number and e is a positive integer. Let $P_1^{(0)}, \dots, P_{q^2+q+1}^{(0)}$ be the \mathbb{F}_q -rational points on $\mathbb{P}_{\mathbb{F}_q}^2$, and let $L_1^{(0)}, \dots, L_{q^2+q+1}^{(0)}$ be the \mathbb{F}_q -lines on $\mathbb{P}_{\mathbb{F}_q}^2$, i.e. the lines which are defined over \mathbb{F}_q . Let

$$f^{(0)}: X^{(0)} \rightarrow \mathbb{P}_{\mathbb{F}_q}^2$$

be the blowup along all the \mathbb{F}_q -points $P_1^{(0)}, \dots, P_{q^2+q+1}^{(0)}$. For any $i = 1, \dots, q^2 + q + 1$, let $E_i^{(0)}$ be the $f^{(0)}$ -exceptional prime divisor lying over $P_i^{(0)}$, hence $E_i^{(0)} \simeq \mathbb{P}_{\mathbb{F}_q}^1$. The proper transforms $L_1'^{(0)}, \dots, L_{q^2+q+1}'^{(0)}$ of the \mathbb{F}_q -lines are disjoint each other and satisfy $(L_i'^{(0)})^2 = -q$ for any $i = 1, \dots, q^2 + q + 1$. Let

$$g^{(0)}: X^{(0)} \rightarrow Y^{(0)}$$

be the birational morphism contracting all of the curves

$$L_1'^{(0)}, \dots, L_{q^2+q+1}'^{(0)}.$$

We define $(E_i^Y)^{(0)} := g_*^{(0)} E_i^{(0)}$.

(2) Let k be a field containing \mathbb{F}_q and let

$$f: X \rightarrow \mathbb{P}_k^2, \quad g: X \rightarrow Y$$

be the base changes of $f^{(0)}$ and $g^{(0)}$ induced by $(-) \times_{\mathbb{F}_q} k$. We denote by P_i, L_i, E_i, L'_i and E_i^Y the inverse images of $P_i^{(0)}, L_i^{(0)}, E_i^{(0)}, L_i'^{(0)}$ and $(E_i^Y)^{(0)}$, respectively. We fix an arbitrary line $H \in |\mathcal{O}_{\mathbb{P}^2}(1)|$ defined over k . By abuse of notation, each P_i (resp. L_j) is also called an \mathbb{F}_q -point (resp. an \mathbb{F}_q -line), although these depend on the choice of the homogeneous coordinates.

Notation 2.2. We use the same notation as in Notation 2.1 but we assume that $q = 2$, i.e. $p = 2$ and $e = 1$.

Remark 2.3. The configuration of the \mathbb{F}_q -points and the \mathbb{F}_q -lines on $\mathbb{P}_{\mathbb{F}_q}^2$ satisfy the following properties:

- For any \mathbb{F}_q -line L on $\mathbb{P}_{\mathbb{F}_q}^2$, the number of the \mathbb{F}_q -points contained in L is equal to $q + 1$.
- For any \mathbb{F}_q -point P on $\mathbb{P}_{\mathbb{F}_q}^2$, the number of the \mathbb{F}_q -lines passing through P is equal to $q + 1$.

If $q = 2$, then the picture of the configuration is classically known as Fano plane (e.g. see [Pol98, Subsection 3.1.1]).

2.3. Basic properties. We now summarise some basic properties of the surfaces X and Y constructed in Notation 2.1.

Lemma 2.4. *We use Notation 2.1. The following hold:*

- (1) $\rho(Y) = 1$.
- (2) Y is klt.
- (3) Y has at most canonical singularities if and only if $q = 2$.
- (4) If $q > 2$, then K_Y is ample.
- (5) If $q = 2$, then $-K_Y$ is ample.
- (6) If $q = 2$, then $-K_X$ is nef and big.

Proof. (1) follows immediately by the construction.

Further, we have

$$g^*K_Y = K_X + \left(1 - \frac{2}{q}\right) \sum_{i=1}^{q^2+q+1} L'_i.$$

Thus, (2) and (3) hold.

We now show (4) and (5). Since $K_X = f^*K_{\mathbb{P}^2} + \sum_i E_i \sim -3f^*H + \sum_i E_i$ and

$$(q^2 + q + 1)f^*H \sim f^*\left(\sum_{i=1}^{q^2+q+1} L_i\right) = \sum_{i=1}^{q^2+q+1} L'_i + (q + 1) \sum_{i=1}^{q^2+q+1} E_i,$$

we have that

$$\begin{aligned}
(q^2 + q + 1)K_X &\sim -3(q^2 + q + 1)f^*H + (q^2 + q + 1) \sum_{i=1}^{q^2+q+1} E_i \\
&\sim -3 \sum_{i=1}^{q^2+q+1} L'_i + (q^2 - 2q - 2) \sum_{i=1}^{q^2+q+1} E_i.
\end{aligned}$$

Taking the push-forward g_* , we get

$$(q^2 + q + 1)K_Y \sim (q^2 - 2q - 2) \sum_{i=1}^{q^2+q+1} E_i^Y.$$

Therefore, if $q = 2$ (resp. $q > 2$), then $-K_Y$ (resp. K_Y) is ample. Thus, (4) and (5) hold. (6) follows directly from (3) and (5). \square

Lemma 2.5. *We use Notation 2.1. We assume that $k = \mathbb{F}_q$. For any \mathbb{F}_q -point $P_i \in \mathbb{P}_{\mathbb{F}_q}^2(\mathbb{F}_q)$, let $L_{j_1}, \dots, L_{j_{q+1}}$ be the \mathbb{F}_q -lines passing through P_i . Then $\mathbb{P}_{\mathbb{F}_q}^2(\mathbb{F}_q) = L_{j_1}(\mathbb{F}_q) \cup \dots \cup L_{j_{q+1}}(\mathbb{F}_q)$.*

Proof. Since we have $L_{j_\alpha} \cap L_{j_\beta} = P_i$ for any $1 \leq \alpha < \beta \leq q+1$, the claim follows by counting the number of \mathbb{F}_q -rational points (cf. Remark 2.3):

$$\#(L_{j_1} \cup \dots \cup L_{j_{q+1}})(\mathbb{F}_q) = q(q+1) + 1 = q^2 + q + 1 = \mathbb{P}_{\mathbb{F}_q}^2(\mathbb{F}_q).$$

\square

3. COUNTER-EXAMPLES TO KAWAMATA–VIEHWEG VANISHING

In this Section, we construct some counter-examples to Kawamata–Viehweg vanishing on a family of smooth rational surfaces:

Theorem 3.1. *We use Notation 2.1. We consider the following \mathbb{Q} -divisors on X :*

- $\Delta := \frac{q}{q+1} \sum_{i=1}^{q^2+q+1} L'_i$, and
- $B := (q^2 + 1)f^*H - q \sum_{i=1}^{q^2+q+1} E_i$.

Then the following hold:

- (1) (X, Δ) is klt.
- (2) $B - \Delta$ is nef and big.
- (3) $h^1(X, \mathcal{O}_X(K_X + B)) \geq \frac{1}{2}(q^2 - q)$.

In particular, Kawamata–Viehweg vanishing fails on X .

Proof. Since $L'_1, \dots, L'_{q^2+q+1}$ are pairwise disjoint, (1) follows immediately. We now show (2). We have:

$$(q^2 + q + 1)f^*H \sim f^*\left(\sum_{i=1}^{q^2+q+1} L_i\right) = \sum_{i=1}^{q^2+q+1} L'_i + (q+1) \sum_{i=1}^{q^2+q+1} E_i.$$

It follows that

$$B = (q^2 + 1)f^*H - q \sum_{i=1}^{q^2+q+1} E_i \sim_{\mathbb{Q}} \frac{1}{q+1}f^*H + \frac{q}{q+1} \sum_{i=1}^{q^2+q+1} L'_i.$$

Thus, (2) holds.

We now show (3). By Riemann–Roch, it follows that

$$\chi(X, \mathcal{O}_X(K_X + B)) = 1 + \frac{1}{2}(B^2 + B \cdot K_X).$$

Since

$$B^2 = ((q^2+1)f^*H - q \sum_{i=1}^{q^2+q+1} E_i)^2 = (q^2+1)^2 - q^2(q^2+q+1) = -q^3 + q^2 + 1$$

and

$$\begin{aligned} B \cdot K_X &= ((q^2+1)f^*H - q \sum_{i=1}^{q^2+q+1} E_i) \cdot (-3f^*H + \sum_{i=1}^{q^2+q+1} E_i) \\ &= -3(q^2+1) + q(q^2+q+1) = q^3 - 2q^2 + q - 3, \end{aligned}$$

we have that

$$\chi(X, K_X + B) = 1 + \frac{1}{2}((-q^3 + q^2 + 1) + (q^3 - 2q^2 + q - 3)) = \frac{1}{2}(-q^2 + q).$$

Thus, (3) holds. \square

Remark 3.2. We do not know whether there exist a klt del Pezzo surface X and a nef and big Cartier divisor A on X such that

$$H^1(X, \mathcal{O}_X(A)) \neq 0.$$

As an application, we now show that the pair $(X, \sum E_i + \sum L'_j)$ is not liftable to $W_2(k)$. Note that, a similar result was proven in [Lan16, Proposition 8.4].

Corollary 3.3. *We use Notation 2.1. Assume that k is perfect. If $p \geq 3$, then*

$$(X, \sum_{i=1}^{q^2+q+1} E_i + \sum_{j=1}^{q^2+q+1} L'_j)$$

is not liftable to $W_2(k)$.

Proof. We use the same notation as in Theorem 3.1. As in the proof of Theorem 3.1, it follows that $B - \Delta - \sum \epsilon_i E_i$ is ample for some $\epsilon_i > 0$. Thus, Theorem 3.1 and [Har98, Corollary 3.8] imply the claim. \square

4. PURELY INSEPARABLE MORPHISMS TO \mathbb{P}^2

The main purpose of this Section is to show that the surface Y , as in Notation 2.1, can be obtained as a purely inseparable cover of \mathbb{P}^2 (cf. Theorem 4.1). Moreover if $q = 2$, then the morphism $Y \rightarrow \mathbb{P}^2$ is induced by the anti-canonical linear system (cf. Remark 4.2).

We also show that the complete linear system $|M|$, appearing in Theorem 4.1, does not have any smooth element (cf. Proposition 4.3), even though it is base point free and big. We were not able to find a similar example in the literature (cf. [Har77, Theorem II.8.18 and Corollary III.10.9]).

Theorem 4.1. *We use Notation 2.1. Let*

$$M := (q+1)f^*H - \sum_{i=1}^{q^2+q+1} E_i.$$

Then the following hold:

- (1) $|M|$ is base point free.
- (2) $M \cdot L'_j = 0$ for any $j = 1, \dots, q^2 + q + 1$.
- (3) $M^2 = q$.
- (4) *Given the natural injective k -linear map*

$$\iota: H^0(X, \mathcal{O}_X(M)) \hookrightarrow H^0(\mathbb{P}_k^2, \mathcal{O}_{\mathbb{P}_k^2}(q+1)),$$

the following holds:

$$\iota(H^0(X, \mathcal{O}_X(M))) = k \cdot (x^q y - x y^q) + k \cdot (y^q z - y z^q) + k \cdot (z^q x - z x^q).$$

- (5) *There exists a Cartier divisor M_Y on Y such that $M = g^*M_Y$.*
- (6) *The morphism induced by the complete linear system $|M_Y|$*

$$\varphi := \Phi_{|M_Y|}: Y \rightarrow \mathbb{P}_k^2$$

is a finite universal homeomorphism of degree q .

Proof. We may assume that $k = \mathbb{F}_q$.

We first show (1). Given a \mathbb{F}_q -point P_i on $\mathbb{P}_{\mathbb{F}_q}^2$, we denote by $L_{j_1}, \dots, L_{j_{q+1}}$ the \mathbb{F}_q -lines passing through P_i . Then Lemma 2.5 implies that

$$M = (q+1)f^*H - \sum_{r=1}^{q^2+q+1} E_r \sim \sum_{\alpha=1}^{q+1} f^*L_{j_\alpha} - \sum_{r=1}^{q^2+q+1} E_r = qE_i + \sum_{\alpha=1}^{q+1} L'_{j_\alpha}.$$

Thus, $|M|$ is base point free by symmetry and (1) holds.

(2) and (3) are simple calculations, and (4) follows from [Tal61a, Tal61b] (see also [HK13, Proposition 2.1]¹). Further, $g: X \rightarrow Y$ is the Stein factorisation of $\psi := \Phi_{|M|}: X \rightarrow \mathbb{P}_k^2$. Thus, (5) holds.

We now show (6). Since $M = g^*M_Y$, (1) implies that $|M_Y|$ is base point free and (5) implies that $h^0(Y, \mathcal{O}_Y(M_Y)) = 3$. Since M_Y is ample, it follows that φ is a finite surjective morphism. By (3), the degree of φ is equal to q .

It is enough to show that φ is a purely inseparable morphism. To this end, we may assume that $k = \overline{\mathbb{F}}_q$. By (4), we have that

$$\psi \circ f^{-1}: \mathbb{P}_k^2 \dashrightarrow \mathbb{P}_k^2, \quad [x : y : z] \mapsto [x^q y - xy^q : y^q z - yz^q : z^q x - zx^q].$$

Generically, the rational map $\psi \circ f^{-1}$ can be written by

$$\Psi: \mathbb{A}_k^2 \setminus \bigcup_{i=1}^{q+1} \tilde{L}_i \rightarrow \mathbb{A}_k^2, \quad (u, v) \mapsto \left(\frac{v^q - v}{u^q v - uv^q}, \frac{u - u^q}{u^q v - uv^q} \right),$$

where $\tilde{L}_1, \dots, \tilde{L}_{q+1}$ are the affine lines passing through the origin with coefficients in \mathbb{F}_q , and in particular $\bigcup_{i=1}^{q+1} \tilde{L}_i = \{u^q v - uv^q = 0\}$. Fix a general closed point $(\alpha, \beta) \in \mathbb{A}_k^2$. It is enough to show that its fibre $\Psi^{-1}((\alpha, \beta))$ consists of one point. Let $(u, v) \in \mathbb{A}_k^2 \setminus \bigcup_{i=1}^{q+1} \tilde{L}_i$ be such that $\Psi(u, v) = (\alpha, \beta)$. Since (α, β) is chosen to be general, we can assume that the denominators of the fractions appearing in the following calculation are always nonzero. We have that

$$\alpha(u^q v - uv^q) = v^q - v, \quad \beta(u^q v - uv^q) = u - u^q,$$

which implies

$$(4.1.1) \quad \alpha(u^q - uv^{q-1}) = v^{q-1} - 1,$$

and

$$(4.1.2) \quad \beta(u^{q-1}v - v^q) = 1 - u^{q-1}.$$

By (4.1.1), we have that

$$(4.1.3) \quad v^{q-1} = \frac{\alpha u^q + 1}{\alpha u + 1}.$$

Substituting (4.1.3) to (4.1.2), we get

$$(4.1.4) \quad v = \frac{1}{\beta} \cdot \frac{1 - u^{q-1}}{u^{q-1} - v^{q-1}} = \frac{1}{\beta} \cdot \frac{1 - u^{q-1}}{u^{q-1} - \frac{\alpha u^q + 1}{\alpha u + 1}} = -\frac{\alpha u + 1}{\beta}.$$

¹Note that we cite the arXiv version, as the published version omits the proof of [HK13, Proposition 2.1].

Substituting (4.1.4) to (4.1.3), it follows that

$$\alpha u^q + 1 = (\alpha u + 1)v^{q-1} = (\alpha u + 1) \left(-\frac{\alpha u + 1}{\beta} \right)^{q-1} = \frac{(-1)^{q-1}(\alpha^q u^q + 1)}{\beta^{q-1}},$$

which implies that

$$u^q = \frac{-\beta^{q-1} + (-1)^{q-1}}{\alpha\beta^{q-1} - (-1)^{q-1}\alpha^q}.$$

This implies that u is uniquely determined by (α, β) , and so is v by (4.1.4). Thus, (6) holds. \square

Remark 4.2. Using the same notation as in Theorem 4.1, if $q = 2$, then $M = -K_X$ and $M_Y = -K_Y$. This can be considered as an analogue of the fact that a smooth del Pezzo surface S with $K_S^2 = 2$ is a double cover of \mathbb{P}^2 which is induced by the anti-canonical system $|-K_X|$. Indeed, both X and S are obtained by taking blowups along seven points.

Proposition 4.3. *We use Notation 2.1. Let*

$$M := (q+1)f^*H - \sum_{i=1}^{q^2+q+1} E_i.$$

Then the following hold:

- (1) *If $k = \mathbb{F}_q$, then for any element $D \in |M|$, there exists a unique \mathbb{F}_q -point P_i on $\mathbb{P}_{\mathbb{F}_q}^2$ such that*

$$D = qE_i + \sum_{\alpha=1}^{q+1} L'_{j_\alpha}$$

where $L_{j_1}, \dots, L_{j_{q+1}}$ are the \mathbb{F}_q -lines passing through P_i .

- (2) *If k is an algebraically closed field, then a general member of $|M|$ is integral.*
- (3) *Any element of $|M|$ is not smooth.*

Proof. Note that for each \mathbb{F}_q -point P_i on $\mathbb{P}_{\mathbb{F}_q}^2$, the divisor $D = qE_i + \sum_{\alpha=1}^{q+1} L'_{j_\alpha}$, as in (1), is an element of $|M|$. Thus, there are $q^2 + q + 1$ of such divisors. On the other hand, (4) of Theorem 4.1 implies

$$\#|M| = \frac{q^3 - 1}{q - 1}.$$

Thus, (1) holds (see also [HK13, Proposition 2.3]).

We now show (2) and (3). To this end, we may assume that k is algebraically closed. We set $M_Y := g_*M$. By (1), there exists an irreducible divisor in $|M_Y|$. Thus, any general element of $|M_Y|$ is irreducible.

Since, by Theorem 4.1, $|M_Y|$ is base point free, if $D \in |M|$ is a general element, then D is irreducible. By Theorem 4.1, we may write

$$f_*D = \{\gamma(x^q y - xy^q) + \alpha(y^q z - yz^q) + \beta(z^q x - zx^q) = 0\}$$

for some $(\alpha, \beta, \gamma) \in k^3 \setminus \{(0, 0, 0)\}$. By the Jacobian criterion for smoothness, it follows that $[\alpha^{1/q} : \beta^{1/q} : \gamma^{1/q}]$ is a unique singular point of f_*D . Since f_*D is smooth outside $[\alpha^{1/q} : \beta^{1/q} : \gamma^{1/q}]$, we see that f_*D is reduced. Since α, β, γ are chosen to be general, it follows that $[\alpha^{1/q} : \beta^{1/q} : \gamma^{1/q}]$ is not an \mathbb{F}_q -point. Thus, D is the proper transform of f_*D , hence D is integral. Thus, (2) holds. Since f_*D has a singular point outside $f(\text{Ex}(f))$, it follows that D is not smooth. Thus, (3) holds. \square

5. THE KLEIMANN–MORI CONE

The main result of this Section is Theorem 5.4 which determines the generators of the Kleimann–Mori cone of X as in Notation 2.2. To this end, we classify the curves whose self-intersection numbers are negative (cf. Proposition 5.3).

Lemma 5.1. *We use Notation 2.2. The following hold:*

- (1) *If C is a curve on X which satisfies $C^2 = -1$ and differs from any of E_1, \dots, E_7 , then*

$$\deg(f_*(C)) \leq 3.$$

- (2) *If C is a curve on X with $C^2 = -2$, then*

$$\deg(f_*(C)) \leq 2.$$

Proof. We show (1). We have

$$C \sim af^*\mathcal{O}_{\mathbb{P}^2}(1) + \sum_{i=1}^7 b_i E_i$$

where $a = \deg(f_*(C)) > 0$ and $b_1, \dots, b_7 \in \mathbb{Z}$. Since $q = 2$, Lemma 2.4 implies that C is a (-1) -curve. Thus, we have

$$\begin{aligned} -1 = C^2 &= a^2 - \sum_{i=1}^7 b_i^2 \\ -1 = K_X \cdot C &= (-3f^*H + \sum_{i=1}^7 E_i) \cdot (af^*H + \sum_{i=1}^7 b_i E_i) = -3a - \sum_{i=1}^7 b_i. \end{aligned}$$

By Schwarz's inequality, we obtain

$$(3a - 1)^2 = \left(\sum_{i=1}^7 b_i \right)^2 \leq 7 \sum_{i=1}^7 b_i^2 = 7(a^2 + 1),$$

which implies $a^2 - 3a - 3 \leq 0$. Thus, (1) holds. The proof of (2) is similar. \square

Lemma 5.2. *We use Notation 2.2. Let C be a curve on X such that $C_0 := f(C)$ is a conic or a cubic. Then $C^2 \geq 0$.*

Proof. First, we assume that C_0 is conic. Suppose that C_0 passes through 5 of the \mathbb{F}_2 -points, say P_1, \dots, P_5 . Let us derive a contradiction. Let P_6 and P_7 be the remaining two \mathbb{F}_2 -points. Since there are exactly three \mathbb{F}_2 -lines passing through P_6 (resp. P_7), we can find an \mathbb{F}_2 -line L_i such that $P_6 \notin L_i$ and $P_7 \notin L_i$. In particular, $C_0 \cap L_i$ contains at least three points, within P_1, \dots, P_5 . This contradicts the fact that $C_0 \cdot L_i = 2$.

Now, we assume that C_0 is cubic. If C_0 is smooth, then $C^2 \geq C_0^2 - 7 = 2$. Thus, we may assume that C_0 is singular and $C^2 < 0$. It follows that C_0 must pass through all the \mathbb{F}_2 -points P_1, \dots, P_7 and the unique singular point of C_0 is an \mathbb{F}_2 -point, say P_1 . Let L_j be an \mathbb{F}_2 -line passing through P_1 . Since $C_0 \cap L_j$ contains at least three \mathbb{F}_2 -rational points $P_1, P_i, P_{i'}$, we have that $C_0 \cdot L_j \geq 4$. This contradicts the fact that $C_0 \cdot L_j = 3$. Thus, the claim follows. \square

Proposition 5.3. *We use Notation 2.2. Let C be a curve on X with $C^2 < 0$. Then C is equal to one of the curves $E_1, \dots, E_7, L'_1, \dots, L'_7$.*

Proof. Assume that $C \notin \{E_1, \dots, E_7\}$. Let $C_0 := f_*C$. Since $-K_X$ is nef and big, we have that $C^2 \geq -2$. Lemma 5.1 implies that $\deg C_0 \leq 3$. By Lemma 5.2, we have that $\deg C_0 = 1$, hence C_0 is a line. Then C_0 passes through at least two of the \mathbb{F}_2 -points. It follows that C_0 is equal to some L_i , hence $C = L'_i$, as desired. \square

Theorem 5.4. *We use Notation 2.2. Then*

$$\overline{NE}(X) = NE(X) = \sum_{i=1}^7 \mathbb{R}_{\geq 0}[E_i] + \sum_{j=1}^7 \mathbb{R}_{\geq 0}[L'_j].$$

Proof. Since there exists an effective \mathbb{Q} -divisor Δ such that (X, Δ) is klt and $-(K_X + \Delta)$ is ample, the cone theorem [Tan16, Theorem 1.7] implies that $NE(X)$ is closed and generated by the extremal rays spanned by curves. By [Tan16a, Theorem 4.3], any extremal ray of $NE(X)$ is generated by a curve C whose self-intersection number is negative. Thus, the claim follows from Proposition 5.3. \square

6. RELATION TO KEEL–M^cKERNAN SURFACES

The goal of this Section is to prove Proposition 6.4 which shows that the surface X , constructed in Notation 2.2, is isomorphic to some surface obtained by Keel–M^cKernan [KM99, end of §9].

We first recall their construction. Let k be a field of characteristic two. We fix a k -rational point in \mathbb{P}_k^2 and a conic over k as follows:

$$Q := [0 : 0 : 1] \in \mathbb{P}_k^2, \quad C := \{xy + z^2 = 0\} \subset \mathbb{P}_k^2.$$

Note that any line through Q is tangent to C . Let $\varphi_0: S_0 \rightarrow \mathbb{P}_k^2$ be the blowup at Q . We choose k -rational points P_1, \dots, P_d at $\varphi_0^{-1}(C)$. We first consider the blowup along these points $\psi: S'_0 \rightarrow S_0$ and then we take the blowup $S \rightarrow S'_0$ along the intersection $\text{Ex}(\psi) \cap \psi_*^{-1}(\varphi_0^{-1}(C))$, where $\psi_*^{-1}(\varphi_0^{-1}(C))$ is the proper transform of $\varphi_0^{-1}(C)$. Note that the intersection $\text{Ex}(\psi) \cap \psi_*^{-1}(\varphi_0^{-1}(C))$ is a collection of k -rational points. We call S a *Keel–M^cKernan surface* of degree d over k .

Let us recall a well-known result on the theory of Severi–Brauer varieties.

Lemma 6.1. *Let X be a projective scheme over \mathbb{F}_q . Let $\overline{\mathbb{F}}_q$ be the algebraic closure of \mathbb{F}_q . If the base change $X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$ is $\overline{\mathbb{F}}_q$ -isomorphic to $\mathbb{P}_{\overline{\mathbb{F}}_q}^n$, then X is \mathbb{F}_q -isomorphic to $\mathbb{P}_{\mathbb{F}_q}^n$.*

Proof. See, for example, [Ser79, Chap. X, §5, §6, §7]. As an alternative proof, one can conclude the claim from [Esn03, Corollary 1.2] and Châtelet’s theorem [GS06, Theorem 5.1.3]. \square

The following two lemmas may be well-known, however we include proofs for the sake of completeness.

Lemma 6.2. *Let k be a field. Take k -rational points*

$$P_1, \dots, P_4, Q_1, \dots, Q_4 \in \mathbb{P}_k^2.$$

Assume that no three of P_1, \dots, P_4 (resp. Q_1, \dots, Q_4) lie on a single line of \mathbb{P}_k^2 . Then there exists a k -automorphism $\sigma: \mathbb{P}_k^2 \rightarrow \mathbb{P}_k^2$ such that $\sigma(P_i) = Q_i$ for any $i \in \{1, 2, 3, 4\}$.

Proof. We may assume that

$$P_1 = [1 : 0 : 0], \quad P_2 = [0 : 1 : 0], \quad P_3 = [0 : 0 : 1], \quad P_4 = [1 : 1 : 1].$$

For each $i \in \{1, 2, 3, 4\}$, we write $Q_i = [a_i : b_i : c_i]$ for some $a_i, b_i, c_i \in k$. Consider the matrix

$$M := \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

Since Q_1, Q_2, Q_3 do not lie on a line, it follows that $\det M \neq 0$. Let $\tau: \mathbb{P}_k^2 \rightarrow \mathbb{P}_k^2$ be the k -automorphism induced by M . In particular,

$$\tau([1 : 0 : 0]) = Q_1, \quad \tau([0 : 1 : 0]) = Q_2, \quad \tau([0 : 0 : 1]) = Q_3.$$

We may write $\tau^{-1}(Q_4) = [d : e : f]$ for some $d, e, f \in k$. Again by the assumption, we have that $d, e, f \neq 0$. Then the k -automorphism

$$\rho: \mathbb{P}_k^2 \rightarrow \mathbb{P}_k^2, \quad [x : y : z] \mapsto [dx : ey : fz]$$

satisfies

$$\begin{aligned} \rho([1 : 0 : 0]) &= [1 : 0 : 0], & \rho([0 : 1 : 0]) &= [0 : 1 : 0], \\ \rho([0 : 0 : 1]) &= [0 : 0 : 1], & \rho([1 : 1 : 1]) &= [d : e : f]. \end{aligned}$$

Therefore, the k -automorphism $\sigma := \tau \circ \rho$ satisfies $\sigma(P_i) = Q_i$ for any $i \in \{1, 2, 3, 4\}$. \square

Lemma 6.3. *Let k be a field of characteristic two. Let C_1 and C_2 be smooth conics in \mathbb{P}_k^2 . Assume that there exist distinct four k -rational points P_1, P_2, P_3, Q of \mathbb{P}_k^2 such that $\{P_1, P_2, P_3\} \subset C_1 \cap C_2$ and the tangent line T_{C_i, P_j} of C_i at P_j passes through Q for any $i \in \{1, 2\}$ and $j \in \{1, 2, 3\}$. Then $C_1 = C_2$.*

Proof. By Lemma 6.2, we may assume that

$$P_1 = [1 : 0 : 0], \quad P_2 = [0 : 1 : 0], \quad P_3 = [1 : 1 : 1], \quad Q = [0 : 0 : 1].$$

It is well-known that C_1 and C_2 are strange curves (e.g. see [Fur14, Theorem 1.1]). [Fur14, Proposition 2.1] implies that for each $i \in \{1, 2\}$, C_i is defined by a quadric homogeneous polynomial:

$$a_i x^2 + b_i xy + c_i y^2 + d_i z^2 \in k[x, y, z].$$

Since $P_1, P_2, P_3 \in C_i$, we get $a_i = c_i = 0$ and $b_i = d_i$. In particular, both of C_1 and C_2 are defined by the same polynomial $xy + z^2$. \square

Proposition 6.4. *Let k be a field of characteristic two. Then any Keel–M^cKernan surface S of degree 3 over k is k -isomorphic to the surface X constructed in Notation 2.2.*

Proof. We use the same notation as above. Let

$$\pi: S_0 \rightarrow \mathbb{P}^1$$

be the induced \mathbb{P}^1 -fibration. We divide the proof into two steps.

Step 1. In this step, we show that any two Keel–M^cKernan surfaces S and S' of degree 3 over k are isomorphic over k .

There are three k -rational points $P_1, P_2, P_3 \in C$ (resp. $P'_1, P'_2, P'_3 \in C$) such that S (resp. S') is the blowup of S_0 twice along $P_1 \cup P_2 \cup P_3$

(resp. $P'_1 \cup P'_2 \cup P'_3$). Thanks to Lemma 6.2, there is a k -automorphism $\sigma: \mathbb{P}_k^2 \rightarrow \mathbb{P}_k^2$ such that $\sigma(Q) = Q$ and $\sigma(P_i) = P'_i$ for $i = 1, 2$ and 3 . Lemma 6.3 implies that $\sigma(C) = C$ and, in particular, σ induces a k -isomorphism $\tilde{\sigma}: S \xrightarrow{\sim} S'$, as desired.

Step 2. In this step, we assume that $k = \mathbb{F}_2$. Note that C has exactly three \mathbb{F}_2 -rational points:

$$Q_1 := [1 : 0 : 0], \quad Q_2 := [0 : 1 : 0], \quad Q_3 := [1 : 1 : 1].$$

Let

$$P_1 := \varphi_0^{-1}(Q_1), \quad P_2 := \varphi_0^{-1}(Q_2), \quad P_3 := \varphi_0^{-1}(Q_3),$$

and S to be the Keel–McKernan surface of degree 3 over \mathbb{F}_2 as above. We now show that S is \mathbb{F}_2 -isomorphic to $X^{(0)}$ defined in Notation 2.2.

There are pairwise disjoint (-1) -curves E_1, \dots, E_7 on S over \mathbb{F}_2 , i.e. for any $i = 1, \dots, 7$, E_i is \mathbb{F}_2 -isomorphic to $\mathbb{P}_{\mathbb{F}_2}^1$ and satisfies $K_S \cdot E_i = E_i^2 = -1$. Indeed, we can check that the following seven curves listed below satisfy these properties.

- The exceptional curve over Q is a (-1) -curve over \mathbb{F}_2 ,
- For any $i = 1, 2, 3$, the exceptional curve over Q_i obtained by the second blowup is a (-1) -curve over \mathbb{F}_2 , and
- For any $1 \leq i < j \leq 3$, the proper transform of the \mathbb{F}_2 -line, passing through Q_i and Q_j , is a (-1) -curve over \mathbb{F}_2 .

Let $\psi: S \rightarrow T$ be the birational morphism with $\psi_*\mathcal{O}_S = \mathcal{O}_T$ that contracts E_1, \dots, E_7 . Since T is a projective scheme over \mathbb{F}_2 whose base change to $\overline{\mathbb{F}_2}$ is a projective plane, it follows that T is \mathbb{F}_2 -isomorphic to $\mathbb{P}_{\mathbb{F}_2}^2$ by Lemma 6.1. Thus, S is obtained by the blowup along all the \mathbb{F}_2 -rational points of $\mathbb{P}_{\mathbb{F}_2}^2$ which implies $S \simeq X^{(0)}$ (cf. Notation 2.2), as desired.

By Step 1 and Step 2, we are done. \square

APPENDIX A. KAWAMATA–VIEHWEG VANISHING FOR SMOOTH DEL PEZZO SURFACES

By Theorem 3.1, there exists a smooth weak del Pezzo surface of characteristic 2 which violates Kawamata–Viehweg vanishing. We now show that Kawamata–Viehweg vanishing holds on smooth del Pezzo surfaces.

Proposition A.1. *Let k be an algebraically closed field of characteristic $p > 0$. Let X be a smooth projective surface over k such that $-K_X$ is ample and let (X, Δ) be a klt pair for some effective \mathbb{Q} -divisor Δ . Let D be a Cartier divisor such that $D - (K_X + \Delta)$ is nef and big.*

Then $H^i(X, \mathcal{O}_X(D)) = 0$ for $i > 0$.

Proof. After perturbing Δ , we may assume that $D - (K_X + \Delta)$ is ample. We define $A := D - (K_X + \Delta)$. We run a $(\Delta + A)$ -MMP $f : X \rightarrow Y$. Since $-K_X$ is ample, Y is also a smooth del Pezzo surface. Moreover, this MMP can be considered as a $(K_X + \Delta + A)$ -MMP. By Kawamata–Viehweg vanishing theorem for birational morphisms (cf. [Kol13, Theorem 10.4], [Tan15, Theorem 2.12]), it follows that

$$H^i(X, \mathcal{O}_X(D)) \simeq H^i(Y, f_*\mathcal{O}_X(D)) \simeq H^i(Y, \mathcal{O}_Y(f_*D))$$

for any i , where the latter isomorphism follows from the fact that f is obtained by running a D -MMP.

Therefore, after replacing X by Y , we may assume that $\Delta + A$ is nef. Thus, $D - K_X$ is nef and big. In this case, it is well-known that $H^i(X, \mathcal{O}_X(D)) = 0$ (e.g. see [Muk13, Proposition 3.2] or [CT16, Proposition 3.3]). \square

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